Bollobás–Meir Conjecture for the TSP in the Unit Cube Holds Asymptotically

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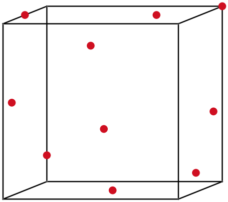
November 18, 2025

Travelling Salesman Problem in the unit cube

find *Hamiltonian cycle* on $X \subseteq [0,1]^k$ with min. $\sum |e|^m$



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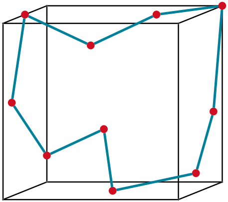

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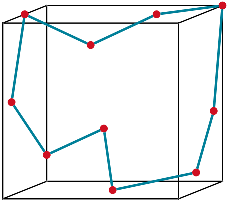

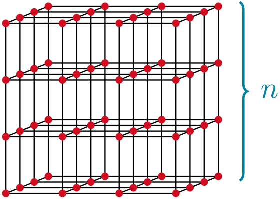

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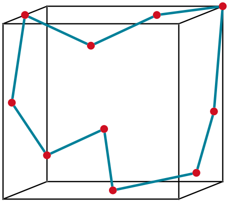

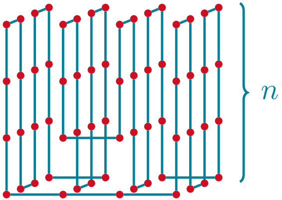

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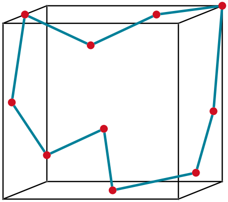
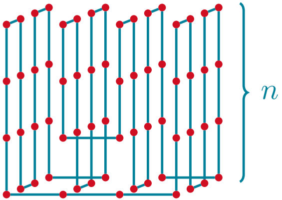
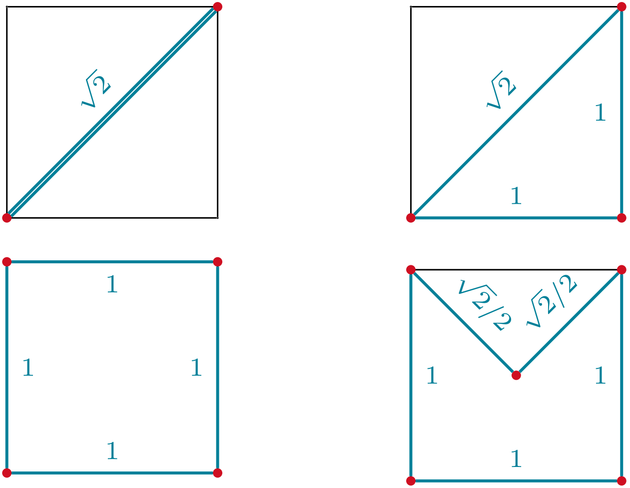




$$\sum |e|^m \approx \frac{n^k}{n^m} \xrightarrow{n\to\infty} \begin{cases} \infty & \text{if } k > m, \\ 0 & \text{if } k < m, \\ \mathbf{1} & \textbf{if } \mathbf{k = m}. \end{cases}$$

? Ham. cycle on $X \subseteq [0,1]^2$ with min. $\sum |e|^2$ **?**



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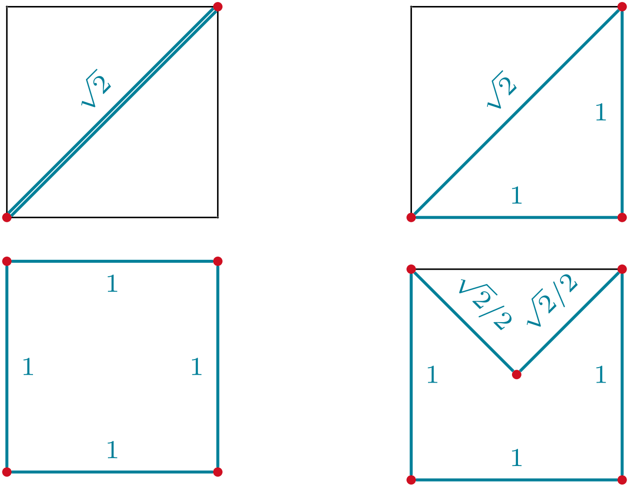
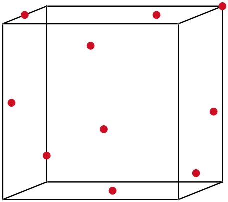


◇ $\forall$ finite $X \subseteq [0,1]^2$ $\exists$ Ham. cycle $\mathbf{H}$ on X: $\sum_{\mathbf{H}} |e|^2 \leq 4$ **Newman 82**

? $\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ on X: $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k} \cdot k^{k/2}$ **?**



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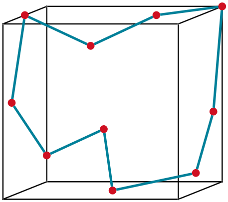

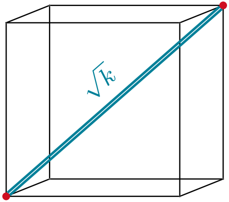

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Bollobás–Meir 93

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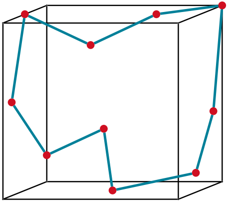

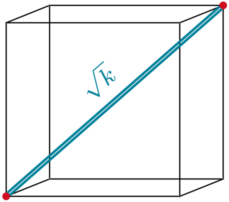

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Bollobás–Meir conjecture

$$\mathbf{c_k} = 2 \quad \forall k \geq 2$$

- $\mathbf{c_2} = 2$ **Newman 82**
- Open for $k > 2$

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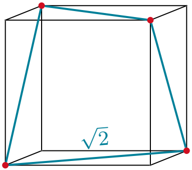
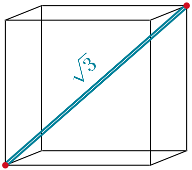


$\mathbf{4(\sqrt{2})^3 > 2(\sqrt{3})^3}$



Bollobás–Meir conjecture (updated BCD 24)

$$\forall \text{ finite } X \subseteq [0,1]^k \ \exists \text{ Ham. cycle } \mathbf{H} \text{ on } X\colon \sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k} \cdot k^{k/2},$$

$$\mathbf{c_k} = 2 \quad \text{for } k \neq 3, \quad \mathbf{c_3} = 4 \cdot \left(\tfrac{2}{3}\right)^{\frac{3}{2}} \approx 2.177$$

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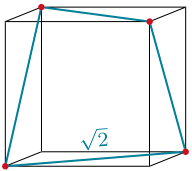

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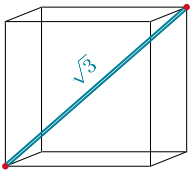

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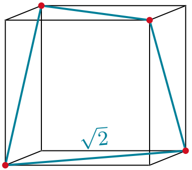

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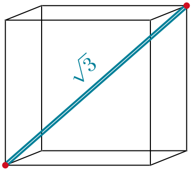

- $\mathbf{c_k} \leq 1.28 \cdot 5.059^k$ or $2.65^k \cdot (1 + o_k(1))$ **G 25**

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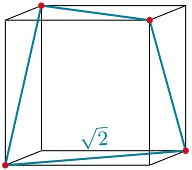

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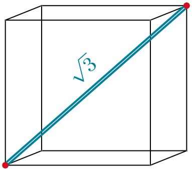

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- **Cycle approximation**: spanning tree $\mathbf{T} \to$ Ham. cycle $\mathbf{H}$

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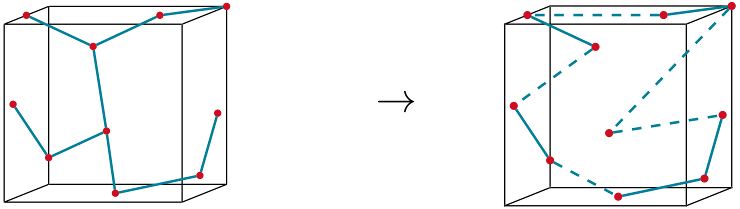

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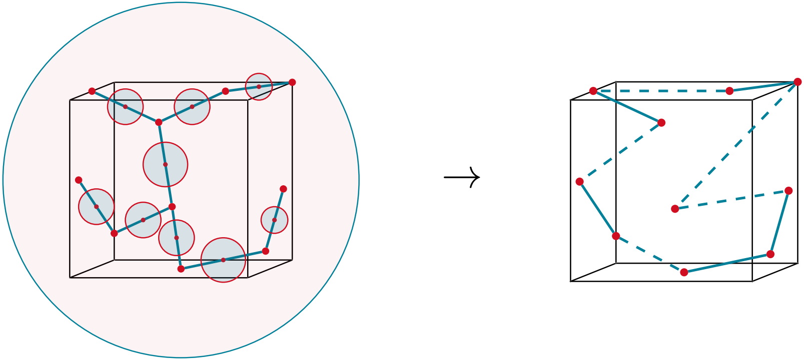
  - volume bound: $\sum_{\mathbf{T}} \left(\frac{|e|}{4}\right)^k \le \left(\frac{3\sqrt{k}}{4}\right)^k \quad \Rightarrow \quad \sum_{\mathbf{T}} |e|^k \le 3^k \cdot k^{k/2}$



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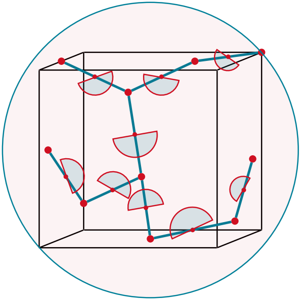

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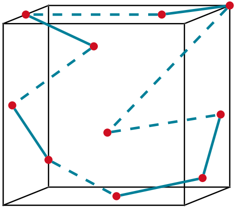

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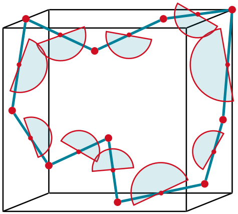

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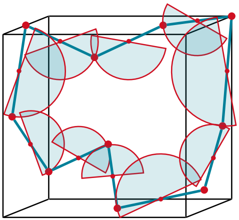

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◇ *+ spherical codes:* $\frac{|e|}{2}$-rad. $3(k+1)$-fold $\mathbf{c_k} \leq 6(k+1)$

◇ *+ centroid properties:* $\sqrt{\frac{t}{t+1}} \cdot \frac{|e|}{2}$-rad. $(2t+1)$-fold $\mathbf{c_k} \leq 2e(k+2)$

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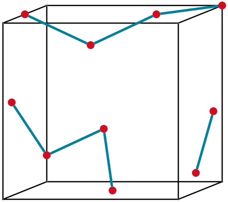

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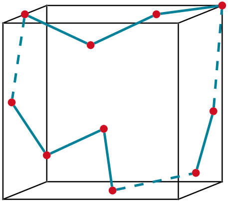

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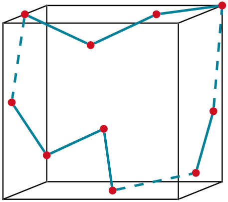

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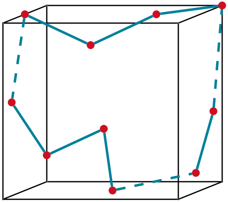
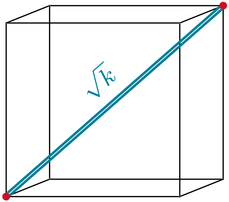




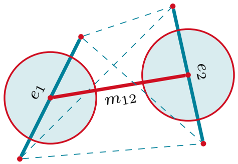
- $\exists \mathbf{H}'$ = collection of disjoint paths $a_{i1} \cdots a_{i2}$ on X:

  $\forall e \subseteq \mathbf{H}' : |e| \leq k^{-1/4}$ and $\forall i \neq j : |a_{is} - a_{jt}| > k^{-1/4}$
- $\mathbf{H}' \rightarrow$ Ham. cycle $\mathbf{H}$: connect paths greedily

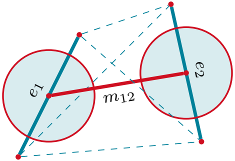
  Ⓐ $\sum_{\mathbf{H} \setminus \mathbf{H}'} |e|^k \leq 2 \cdot k^{k/2} + o_k(k^{k/2})$

  Ⓑ $\sum_{\mathbf{H}'} |e|^k = o_k(k^{k/2})$

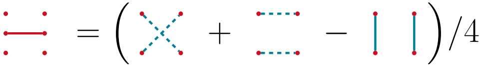
Prove: perfect matching $\mathbf{M}$ with min. $\sum_{\mathbf{M}} |e|^2 \;\Rightarrow\; \frac{|e|}{2\sqrt{2}}$-rad. packing



Prove: perfect matching $\mathbf{M}$ with min. $\sum_{\mathbf{M}} |e|^2 \;\Rightarrow\; \frac{|e|}{2\sqrt{2}}$-rad. packing



$$\forall a,b,c,d \in \mathbb{R}^k : \left|\frac{a+b}{2} - \frac{c+d}{2}\right|^2 = \frac{|a-c|^2+|b-d|^2+|a-d|^2+|b-c|^2-|a-b|^2-|c-d|^2}{4}$$



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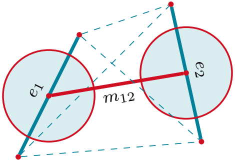

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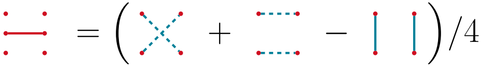

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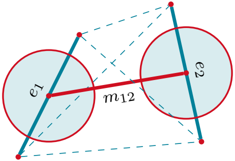

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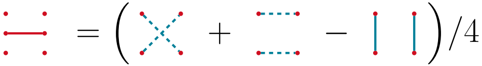
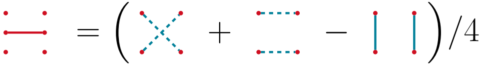


$\mathbf{M}$:

$$|m_{12}|^2 \geq \frac{|e_1|^2+|e_2|^2}{4} \geq \frac{(|e_1|+|e_2|)^2}{8} \quad \Rightarrow \quad |m_{12}| \geq \frac{|e_1|+|e_2|}{2\sqrt{2}}$$

Prove: Ham. cycle $\mathbf{H}$ with min. $\sum_{\mathbf{H}} |e|^2 \quad \Rightarrow \quad$ 3-fold $\frac{|e|}{2\sqrt{2}}$-rad. packing



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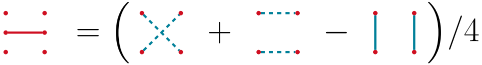

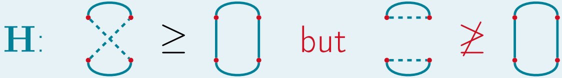

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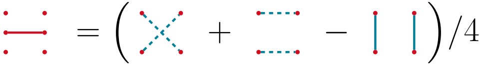

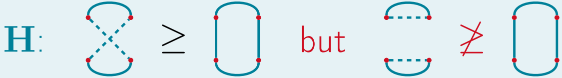

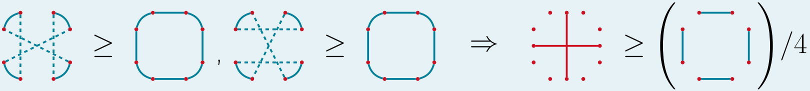

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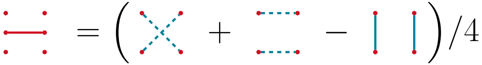

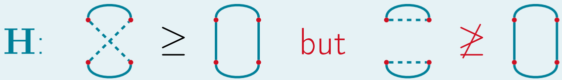

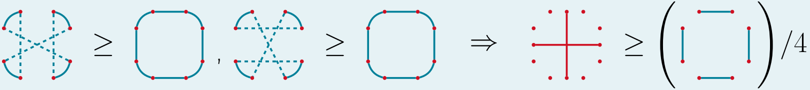

Open questions

Bollobás–Meir conjecture

$\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ on X: $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k} \cdot k^{k/2}$,

$\mathbf{c_k} = 2$ for $k \neq 3$, $\quad \mathbf{c_3} = 4 \cdot (\frac{2}{3})^{\frac{3}{2}} \approx 2.177$

- $\mathbf{c_k} = 2 + o_k(1)$, but conjecture is still open

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- $\mathbf{c_k} = 2 + o_k(1)$, but conjecture is still open

Algorithmic version

Find in poly. time Ham. cycle $\mathbf{H}$ on $X \subseteq [0,1]^k$: $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k^{poly}} \cdot k^{k/2}$,

$$\mathbf{c_k^{poly}} = ?$$

- using **Bender–Chekuri 00** approx. alg.: $\mathbf{c_k^{poly}} \lesssim 2^k \cdot (1 + o_k(1))$

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A different boundary condition on X (asked in BCD 24)

$$\forall \text{ finite } X, \operatorname{diam} X \leq 1, \exists \text{ Ham. cycle } \mathbf{H} \text{ on } X\text{: } \sum_{\mathbf{H}} |e|^k \leq ?$$