Bollobás–Meir TSP conjecture holds asymptotically

Alexey Gordeev

Umeå University, Sweden
HUN-REN Alfréd Rényi Institute of Mathematics, Budapest, Hungary

June 30, 2026

A worst-case TSP

? best $c(n)$: $\forall\ X \subseteq [0,1]^2$, $|X| = n$, $\exists$ Ham. cycle $\mathbf{H}$: $\sum_{\mathbf{H}} |e| \leq c(n)$

A worst-case TSP

? best $c(n)$: $\forall\ X \subseteq [0,1]^2$, $|X| = n$, $\exists$ Ham. cycle $\mathbf{H}$: $\sum_{\mathbf{H}} |e| \leq c(n)$

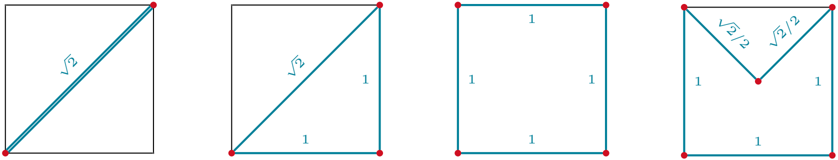
- $c(n) \geq 1.075\sqrt{n} - o(\sqrt{n})$ **Tóth, Few 50s**
- $c(n) \leq 1.392\sqrt{n} + 7/4$ **Karloff 89**
- $X \subseteq [0,1]^k$: $\quad c_k(n) = \Theta(n^{1-1/k})$

A worst-case TSP

? best $c(n)$: $\forall\ X \subseteq [0,1]^2$, $|X| = n$, $\exists$ Ham. cycle $\mathbf{H}$: $\sum_{\mathbf{H}} |e| \leq c(n)$

- $c(n) \geq 1.075\sqrt{n} - o(\sqrt{n})$ **Tóth, Few 50s**
- $c(n) \leq 1.392\sqrt{n} + 7/4$ **Karloff 89**
- $X \subseteq [0,1]^k$: $\quad c_k(n) = \Theta(n^{1-1/k})$

? best $c'(n)$: $\forall\ X \subseteq [0,1]^2$, $|X| = n$, $\exists$ Ham. cycle $\mathbf{H}$: $\sum_{\mathbf{H}} |e|^2 \leq c'(n)$

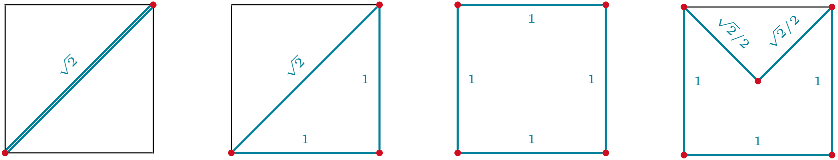


A worst-case TSP

? best $c(n)$: $\forall\ X \subseteq [0,1]^2$, $|X| = n$, $\exists$ Ham. cycle $\mathbf{H}$: $\sum_{\mathbf{H}} |e| \leq c(n)$

- $c(n) \geq 1.075\sqrt{n} - o(\sqrt{n})$ **Tóth, Few 50s**
- $c(n) \leq 1.392\sqrt{n} + 7/4$ **Karloff 89**
- $X \subseteq [0,1]^k$: $\quad c_k(n) = \Theta(n^{1-1/k})$

? best $c'(n)$: $\forall\ X \subseteq [0,1]^2$, $|X| = n$, $\exists$ Ham. cycle $\mathbf{H}$: $\sum_{\mathbf{H}} |e|^2 \leq c'(n)$



- $c'(n) = 4$ **Newman 82**

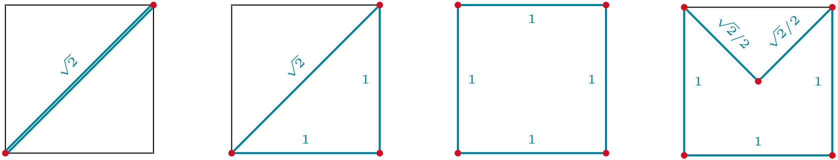
  does not depend on n!

A worst-case TSP

? best $c(n)$: $\forall\ X \subseteq [0,1]^2$, $|X| = n$, $\exists$ Ham. cycle $\mathbf{H}$: $\sum_{\mathbf{H}} |e| \leq c(n)$

- $c(n) \geq 1.075\sqrt{n} - o(\sqrt{n})$ **Tóth, Few 50s**
- $c(n) \leq 1.392\sqrt{n} + 7/4$ **Karloff 89**
- $X \subseteq [0,1]^k$: $\quad c_k(n) = \Theta(n^{1-1/k})$

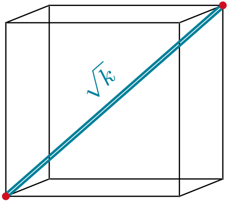
? best $c'(n)$: $\forall\ X \subseteq [0,1]^2$, $|X| = n$, $\exists$ Ham. cycle $\mathbf{H}$: $\sum_{\mathbf{H}} |e|^2 \leq c'(n)$



- $c'(n) = 4$ **Newman 82**

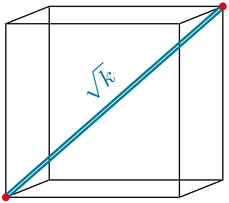
  does not depend on n!
- what if $X \subseteq [0,1]^k$, $k > 2$? **Meir 87**

? best $\mathbf{c_k}$: $\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ with $\sum_{\mathbf{H}} |e|^k \le \mathbf{c_k}$



◇ $2 \cdot k^{k/2} \le \mathbf{c_k} \le \frac{2}{3} \cdot 9^k \cdot k^{k/2}$ **Bollobás–Meir 93**

? best $\mathbf{c_k}$: $\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ with $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k}$



- $2 \cdot k^{k/2} \leq \mathbf{c_k} \leq \frac{2}{3} \cdot 9^k \cdot k^{k/2}$ **Bollobás–Meir 93**

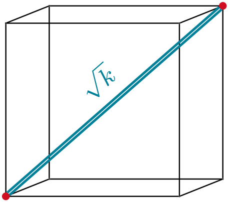
Bollobás–Meir TSP conjecture

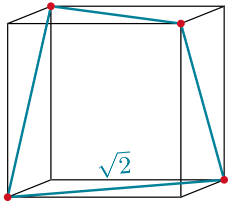
$$\mathbf{c_k} = 2 \cdot k^{k/2} \quad \text{for } k \geq 2$$

- $\mathbf{c_2} = 4$ **Newman 82**

  open for $k > 2$

? best $\mathbf{c_k}$: $\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ with $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k}$





◇ $2 \cdot k^{k/2} \leq \mathbf{c_k} \leq \frac{2}{3} \cdot 9^k \cdot k^{k/2}$ **Bollobás–Meir 93**

Bollobás–Meir TSP conjecture (updated BCD 24)

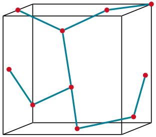
$$\mathbf{c_k} = 2 \cdot k^{k/2} \quad \text{for } k \neq 3, \quad \mathbf{c_3} = 2^{7/2}$$

◇ $\mathbf{c_2} = 4$ **Newman 82**

open for $k > 2$

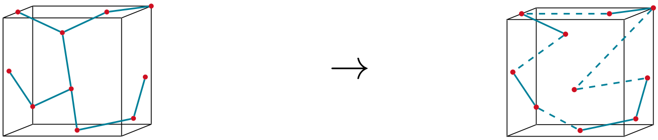
main tools: **cycle approximation** and **ball packing**

- $\mathbf{T}$: min. spanning tree on X



main tools: **cycle approximation** and **ball packing**

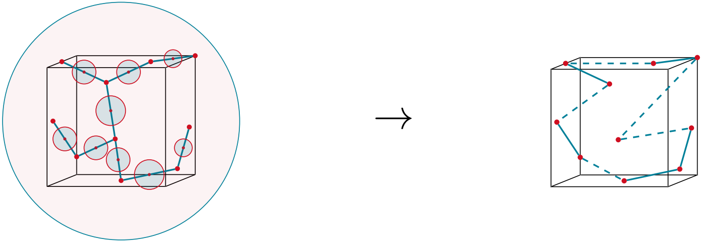
- $\mathbf{T}$: min. spanning tree on X



A $\exists$ Ham. cycle $\mathbf{H}$: $\sum_{\mathbf{H}} |e|^k \leq \frac{2}{3} \cdot 3^k \cdot \sum_{\mathbf{T}} |e|^k$ **Sekanina 60s**

main tools: **cycle approximation** and **ball packing**

- $\mathbf{T}$: min. spanning tree on X

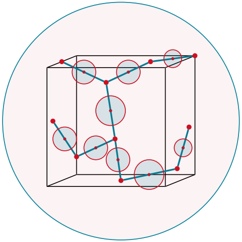


Ⓐ $\exists$ Ham. cycle $\mathbf{H}$: $\sum_{\mathbf{H}} |e|^k \leq \frac{2}{3} \cdot 3^k \cdot \sum_{\mathbf{T}} |e|^k$ **Sekanina 60s**

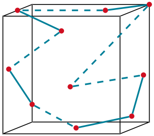
Ⓑ $\frac{|e|}{4}$-rad. balls on edges of $\mathbf{T}$ are disjoint $\rightarrow \sum_{\mathbf{T}} |e|^k \leq 3^k \cdot k^{k/2}$

main tools: **cycle approximation** and **ball packing**

- $\mathbf{T}$: min. spanning tree on X



$\rightarrow$



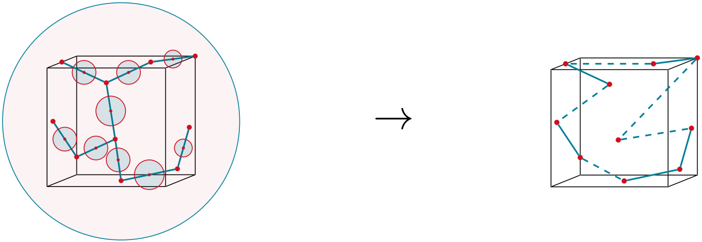
A $\exists$ Ham. cycle $\mathbf{H}$: $\sum_{\mathbf{H}} |e|^k \leq \frac{2}{3} \cdot 3^k \cdot \sum_{\mathbf{T}} |e|^k$ **Sekanina 60s**

B $\frac{|e|}{4}$-rad. balls on edges of $\mathbf{T}$ are disjoint $\rightarrow \sum_{\mathbf{T}} |e|^k \leq 3^k \cdot k^{k/2}$

◇ A + B $\rightarrow \mathbf{c_k} \leq \frac{2}{3} \cdot 9^k \cdot k^{k/2}$ **Bollobás–Meir 93**

main tools: **cycle approximation** and **ball packing**

- $\mathbf{T}$: min. spanning tree on X



Ⓐ $\exists$ Ham. cycle $\mathbf{H}$: $\sum_{\mathbf{H}} |e|^k \leq \frac{2}{3} \cdot 3^k \cdot \sum_{\mathbf{T}} |e|^k$ **Sekanina 60s**

Ⓑ $\frac{|e|}{4}$-rad. balls on edges of $\mathbf{T}$ are disjoint $\rightarrow \sum_{\mathbf{T}} |e|^k \leq 3^k \cdot k^{k/2}$

◇ Ⓐ + Ⓑ $\rightarrow \mathbf{c_k} \leq \frac{2}{3} \cdot 9^k \cdot k^{k/2}$ **Bollobás–Meir 93**

◇ $\mathbf{c_k} \leq 1.28 \cdot 5.059^k \cdot k^{k/2}$

$\mathbf{c_k} \leq 2.65^k \cdot k^{k/2} \cdot (1 + o_k(1))$ **Balogh–Clemen–Dumitrescu 24, G 25**

used the same combination of packing and approximation

Bollobás–Meir TSP conjecture (updated BCD 24)

$\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ on X: $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k}$,

$\mathbf{c_k} = 2 \cdot k^{k/2}$ for $k \neq 3$, $\quad \mathbf{c_3} = 2^{7/2}$

- $\mathbf{c_k} \leq 1.28 \cdot 5.059^k \cdot k^{k/2}$ or $2.65^k \cdot k^{k/2} \cdot (1 + o_k(1))$ **BCD 24, G 25**

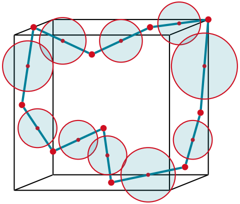
Bollobás–Meir TSP conjecture (updated BCD 24)

$\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ on X: $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k}$,

$\mathbf{c_k} = 2 \cdot k^{k/2}$ for $k \neq 3$, $\mathbf{c_3} = 2^{7/2}$

- $\mathbf{c_k} \leq 1.28 \cdot 5.059^k \cdot k^{k/2}$ or $2.65^k \cdot k^{k/2} \cdot (1 + o_k(1))$ **BCD 24, G 25**

  main new idea: no need to approximate! t-fold packing on $\mathbf{H}$ directly



- $\mathbf{c_k} \leq 2\mathrm{e}(k+1) \cdot k^{k/2}$ **G 26**

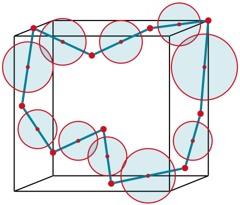
Bollobás–Meir TSP conjecture (updated BCD 24)

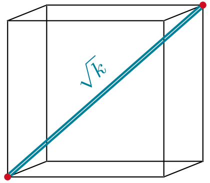
$\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ on X: $\sum_{\mathbf{H}} |e|^k \le \mathbf{c_k}$,

$\mathbf{c_k} = 2 \cdot k^{k/2}$ for $k \neq 3$, $\mathbf{c_3} = 2^{7/2}$

- $\mathbf{c_k} \le 1.28 \cdot 5.059^k \cdot k^{k/2}$ or $2.65^k \cdot k^{k/2} \cdot (1 + o_k(1))$ **BCD 24, G 25**

  main new idea: no need to approximate! t-fold packing on $\mathbf{H}$ directly



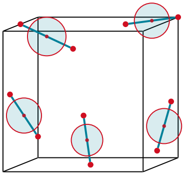


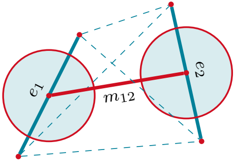
- $\mathbf{c_k} \le 2\mathrm{e}(k+1) \cdot k^{k/2}$ **G 26**
- $\mathbf{c_k} = k^{k/2} \cdot (2 + o_k(1))$ i.e. the conjecture holds asymptotically **G 26**

  $\sum_{\mathbf{H}\setminus\{e_1,e_2\}} |e|^k = o_k(k^{k/2})$, where e_1, e_2 are longest in $\mathbf{H}$

Warm-up

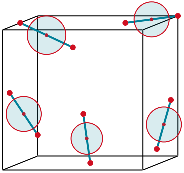
Prove: perfect matching $\mathbf{M}$ with min. $\sum_{\mathbf{M}} |e|^2 \quad \Rightarrow \quad \frac{|e|}{2\sqrt{2}}$-rad. packing

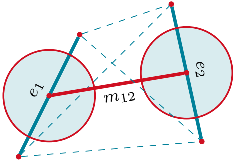




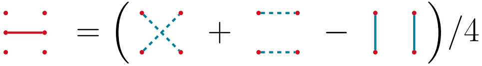
Warm-up

Prove: perfect matching $\mathbf{M}$ with min. $\sum_{\mathbf{M}} |e|^2 \quad \Rightarrow \quad \frac{|e|}{2\sqrt{2}}$-rad. packing



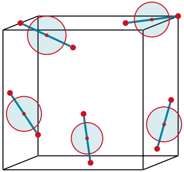


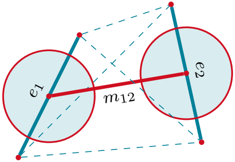
$$\forall a, b, c, d \in \mathbb{R}^k : \left|\frac{a+b}{2} - \frac{c+d}{2}\right|^2 = \frac{|a-c|^2+|b-d|^2+|a-d|^2+|b-c|^2-|a-b|^2-|c-d|^2}{4}$$



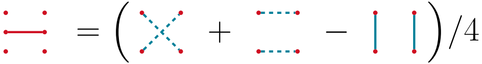
Warm-up

Prove: perfect matching $\mathbf{M}$ with min. $\sum_{\mathbf{M}} |e|^2 \;\Rightarrow\; \frac{|e|}{2\sqrt{2}}$-rad. packing





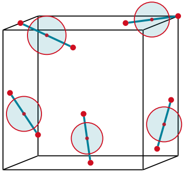
$$\forall a, b, c, d \in \mathbb{R}^k : \left|\frac{a+b}{2} - \frac{c+d}{2}\right|^2 = \frac{|a-c|^2+|b-d|^2+|a-d|^2+|b-c|^2-|a-b|^2-|c-d|^2}{4}$$

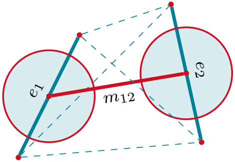


$\mathbf{M}$:

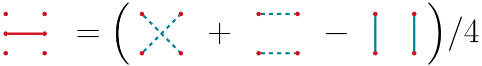
Warm-up

Prove: perfect matching $\mathbf{M}$ with min. $\sum_{\mathbf{M}} |e|^2 \quad \Rightarrow \quad \frac{|e|}{2\sqrt{2}}$-rad. packing





$$\forall a, b, c, d \in \mathbb{R}^k : \left|\frac{a+b}{2} - \frac{c+d}{2}\right|^2 = \frac{|a-c|^2+|b-d|^2+|a-d|^2+|b-c|^2-|a-b|^2-|c-d|^2}{4}$$

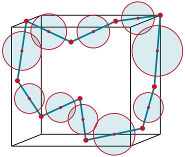


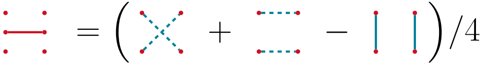
$\mathbf{M}$:

$$|m_{12}|^2 \geq \frac{|e_1|^2+|e_2|^2}{4} \geq \frac{(|e_1|+|e_2|)^2}{8} \quad \Rightarrow \quad |m_{12}| \geq \frac{|e_1|+|e_2|}{2\sqrt{2}}$$

3-fold packing directly on $\mathbf{H}$

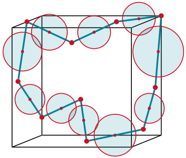
Prove: Ham. cycle $\mathbf{H}$ with min. $\sum_{\mathbf{H}} |e|^2 \quad \Rightarrow \quad$ 3-fold $\frac{|e|}{2\sqrt{2}}$-rad. packing

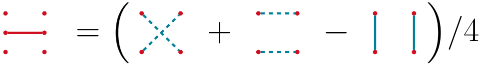


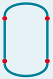
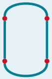


3-fold packing directly on $\mathbf{H}$

Prove: Ham. cycle $\mathbf{H}$ with min. $\sum_{\mathbf{H}} |e|^2 \quad \Rightarrow \quad$ 3-fold $\frac{|e|}{2\sqrt{2}}$-rad. packing

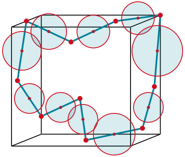


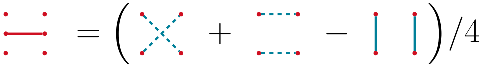


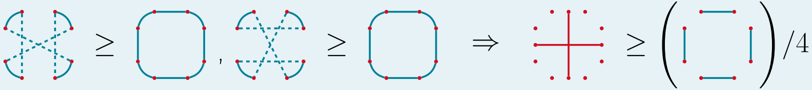
$\mathbf{H}$:  $\geq$  but  $\not\geq$ 

3-fold packing directly on $\mathbf{H}$

Prove: Ham. cycle $\mathbf{H}$ with min. $\sum_{\mathbf{H}} |e|^2 \quad \Rightarrow \quad$ 3-fold $\frac{|e|}{2\sqrt{2}}$-rad. packing

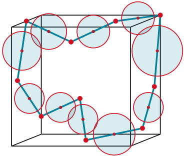


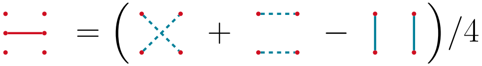


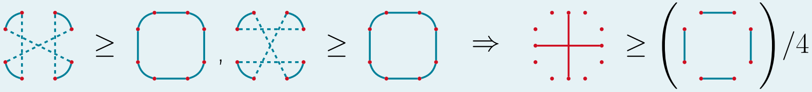


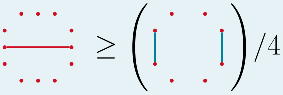
3-fold packing directly on $\mathbf{H}$

Prove: Ham. cycle $\mathbf{H}$ with min. $\sum_{\mathbf{H}} |e|^2 \quad \Rightarrow \quad$ 3-fold $\frac{|e|}{2\sqrt{2}}$-rad. packing

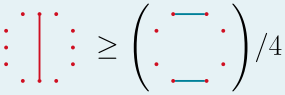






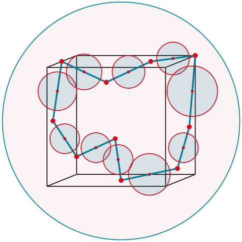


or



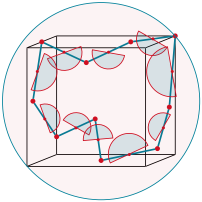
? best $\mathbf{c_k}$: $\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ with $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k}$

- volume bound on $\frac{|e|}{2\sqrt{2}}$-rad. 3-fold packing



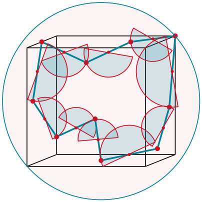
? best $\mathbf{c_k}$: $\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ with $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k}$

◇ volume bound on $\frac{|e|}{2\sqrt{2}}$-rad. 3-fold half-ball packing $\mathbf{c_k} \leq 6 \cdot (\sqrt{2})^k \cdot k^{k/2}$



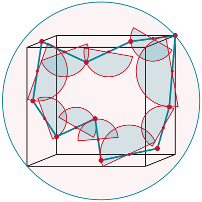
? best $\mathbf{c_k}$: $\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ with $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k}$

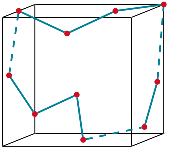
- volume bound on $\frac{|e|}{2\sqrt{2}}$-rad. 3-fold half-ball packing $\mathbf{c_k} \leq 6 \cdot (\sqrt{2})^k \cdot k^{k/2}$
- centroid properties: $\sqrt{\frac{t}{t+1}}\frac{|e|}{2}$-rad. $(2t+1)$-fold $\mathbf{c_k} \leq 2\mathrm{e}(k+1) \cdot k^{k/2}$



? best $\mathbf{c_k}$: $\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ with $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k}$

- volume bound on $\frac{|e|}{2\sqrt{2}}$-rad. 3-fold half-ball packing $\mathbf{c_k} \leq 6 \cdot (\sqrt{2})^k \cdot k^{k/2}$
- centroid properties: $\sqrt{\frac{t}{t+1}}\frac{|e|}{2}$-rad. $(2t+1)$-fold $\mathbf{c_k} \leq 2\mathrm{e}(k+1) \cdot k^{k/2}$



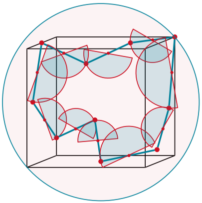


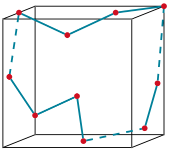
- estimate long edges separately (as **BCD 24** did for spanning trees)

  $\mathbf{H}'$: *short-edge* paths *far* from each other $\mathbf{H}' \to \mathbf{H}$: connect greedily

? best $\mathbf{c_k}$: $\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ with $\sum_{\mathbf{H}} |e|^k \le \mathbf{c_k}$

- volume bound on $\frac{|e|}{2\sqrt{2}}$-rad. 3-fold half-ball packing $\mathbf{c_k} \le 6 \cdot (\sqrt{2})^k \cdot k^{k/2}$
- centroid properties: $\sqrt{\frac{t}{t+1}} \frac{|e|}{2}$-rad. $(2t+1)$-fold $\mathbf{c_k} \le 2\mathrm{e}(k+1) \cdot k^{k/2}$





- estimate long edges separately (as **BCD 24** did for spanning trees)

  $\mathbf{H'}$: *short-edge* paths *far* from each other $\mathbf{H'} \to \mathbf{H}$: connect greedily

  volume bound on $\mathbf{H'}$ + # (long edges) is small $\mathbf{c_k} = k^{k/2} \cdot (2 + o_k(1))$

Open questions

Bollobás–Meir TSP conjecture (updated BCD 24)

$$\forall \text{ finite } X \subseteq [0,1]^k \ \exists \text{ Ham. cycle } \mathbf{H} \text{ on } X\text{: } \sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k},$$

$$\mathbf{c_k} = 2 \cdot k^{k/2} \text{ for } k \neq 3, \quad \mathbf{c_3} = 2^{7/2}$$

- $\mathbf{c_k} = k^{k/2} \cdot (2 + o_k(1))$, but the conjecture is still open

Open questions

Bollobás–Meir TSP conjecture (updated BCD 24)

$\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle $\mathbf{H}$ on X: $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k}$,

$\mathbf{c_k} = 2 \cdot k^{k/2}$ for $k \neq 3$, $\quad \mathbf{c_3} = 2^{7/2}$

- $\mathbf{c_k} = k^{k/2} \cdot (2 + o_k(1))$, but the conjecture is still open

Algorithmic version

Find in polynomial time Ham. cycle $\mathbf{H}$ on $X \subseteq [0,1]^k$: $\sum_{\mathbf{H}} |e|^k \leq \mathbf{c_k^{poly}}$,

best $\mathbf{c_k^{poly}} = ?$

- using **Bender–Chekuri 00** approx. alg.: $\mathbf{c_k^{poly}} \leq 2^{k+2} \cdot k^{k/2} \cdot (1 + o_k(1))$

Open questions

Bollobás–Meir TSP conjecture (updated BCD 24)

$$\forall \text{ finite } X \subseteq [0,1]^k \ \exists \text{ Ham. cycle } \mathbf{H} \text{ on } X\text{: } \sum_{\mathbf{H}} |e|^k \le \mathbf{c_k},$$
$$\mathbf{c_k} = 2 \cdot k^{k/2} \text{ for } k \neq 3, \quad \mathbf{c_3} = 2^{7/2}$$

- $\mathbf{c_k} = k^{k/2} \cdot (2 + o_k(1))$, but the conjecture is still open

Algorithmic version

Find in polynomial time Ham. cycle $\mathbf{H}$ on $X \subseteq [0,1]^k$: $\sum_{\mathbf{H}} |e|^k \le \mathbf{c_k^{poly}}$,

best $\mathbf{c_k^{poly}} = ?$

- using **Bender–Chekuri 00** approx. alg.: $\mathbf{c_k^{poly}} \le 2^{k+2} \cdot k^{k/2} \cdot (1 + o_k(1))$

A different boundary condition on X (asked in BCD 24)

$$\forall \text{ finite } X,\ \operatorname{diam} X \le 1,\ \exists \text{ Ham. cycle } \mathbf{H} \text{ on } X\text{: } \sum_{\mathbf{H}} |e|^k \le \mathbf{c'_k},$$

best $\mathbf{c'_k} = ?$

- $X =$ simplex: $\mathbf{c'_k} \ge k+1$ present methods: $\mathbf{c'_k} \le 2\mathrm{e} \cdot 2^{k/2} \cdot (k+1)$